A ONE-EIGHTH REPLICATE OF A 2^9 FACTORIAL

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A marketing survey was conducted on the movement of apples in retail grocery stores of western New York during the fall and winter of 1952-53. In order to obtain some idea of the relative variability of certain kinds of stratification it was decided to study two sizes of grocery store, four days of the week, four times of the day, four sizes of city, and four areas. All combinations of the variables would have resulted in a $2x4^4 = 512$ factorial arrangement. Since it was not possible to carry out all 512 combinations the 64 combinations presented in figure 1 were selected. These combinations (figure 1) were renumbered as a 2^9 factorial instead of as a $2x4^4$ factorial (see figure 2). From the renumbered treatments it was possible to determine the fraction of the replicate obtained (see O. Kempthorne, Biometrika 34: 255). A study of figures 1 and 2 shows that no main effects are confounded with each other and that the 64 selected treatments as numbered do not form a complete fractional replicate in the ordinary sense. It may be possible to renumber the treatments and to obtain the 64 treatments forming a one-eighth replicate but this was not attempted because an analysis of these combinations is possible. If the 16 numbers in parentheses in figure 2 are used to replace their counterparts a one-eighth replicate is obtained.

The defining contrast equal to I = BEF = ADFH = ABDEH = CDFG = BCDEG = ACGH = ABCEFGH results in the 64 combinations necessary for a one-eighth

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Considerable aid in developing the analysis for this experiment was obtained from Dr. Max E. Brunk and Mr. Martin A. Blum. Dr. Brunk built a special device for obtaining the 32 treatments for each level of an effect in the one-eighth replicate of the 2^9 factorial. The gadget was very useful in determining the fractional replicate used. The analyses and interpretation of the data will form the basis for Mr. Blum's Ph.D. thesis.

replicate. The complete set of aliases for the 64 effects is given in figure 3. For each of the 504 effects not in the defining contrast (figure 3) and for a one-eighth replicate 32 of the 64 treatments should make up the zero level (modulo 2) of the effect, and the other 32 should make up the one level (modulo 2) of the effect. All of the effects in the defining contrast should make up either the zero or the one level (modulo 2) of the effect. Due to the particular set of 64 treatments selected it was found that the following 16 effects were in the ratio 16:48 instead of 32:32:

ACGH	ABCGH	ACEGH	ABCEGH
CDFG	ACEFGH	ABCFGH	BCDEFG
BCDEG	BCDFG	CDEFG	ACFGH
ABCEFGH	CDEG	BCDG	CDG

The four effects in the first column are in the defining contrast; the four effects in the second column are in the contrast for the B effect; the four effects in the third column are in the contrast for the E effect; and the four effects in the fourth column are in the contrast for the F effect. Since none of the above 16 effects was expected to be different from zero (aside from sampling variation) the effect of the disproportionality upon the analysis for this particular one-eighth replicate may be considered negligible.

Therefore, the next step is to decide which effects may be considered as not differing significantly from zero. From previous work it was decided that the following interactions could be expected to have some effect:

Interaction	df
Size of store (a) and size of town (b and c)	3
Size of store (a) and day of week (f and g)	3
Size of store (a) and time of day (h and i)	3
Size of town (b and c) and geographic area (d and e)	9
Size of town (b and c) and time of day (h and i)	9

The sum of the degrees of freedom for interactions and main effects is equal to $(1 + 3 + 3 + 3 + 3) + (3 + 3 + 3 + 9 + 9) = 40$, but due to the fact

that some effects are aliases of others, the sum of the independent contrasts is equal to 13 + (3 + 3 + 3 + 5 + 8) = 35. Since the total number of degrees of freedom is 63, this leaves 28 degrees of freedom associated with the residual sum of squares (figure 4). The residual error mean square may be used for testing the existence of the other main effects and interactions.

However, not all effects in an interaction can be considered as real. Certain components should be partitioned out and relegated to the residual sum of squares for error if they do not appear as an alias of one of the effects. In the above, the four levels of a factor, say size of town, have been considered as a 2x2 factorial with the following comparisons among the levels:

	Level of factor			
Effect	1	2	3	4
B	-1	1	-1	1
C	-1	-1	1	1
BC	1	-1	-1	1

The following comparisons among the levels of a given factor represent the linear, quadratic, and cubic effects (see G. W. Snedecor, Statistical Methods, 4th Ed., page 410):

	Level of factor			
Effect	1	2	3	4
Linear	-3	-1	1	3
Quadratic	1	-1	-1	1
Cubic	-1	3	-3	1

The quadratic effect represents the same comparison as the BC interaction. This is also true for the DE, FG, and HI interactions. Since any interaction with the quadratic effect (BC, DE, FG, and HI) of the variables size of town, area, day of week, and time of day may be expected to have no effect these components may be included in the error mean square unless they happen to be an alias of one of the other effects (see figure 4). The separate components of an interaction are given in figure 4.

Partial information is available on the interactions considered of

some importance. The particular components of interactions which are free of other effects will need to be interpreted in light of the particular factors and levels of the factors being compared.

The analysis for the second part of the week, which includes the shopping periods after four o'clock on each day and Fridays and Saturdays, follows that outlined above for the first part of the week except that the additional interaction of time of day (h and i) and day of week (f and g) is expected.

Only two randomizations of the above design were made. One randomization or pattern was followed for the first part of the week and a second pattern of entering stores was followed for the second part of the week. Therefore, the parts of any given week may be analyzed separately or together, or the totals for all weeks may be analyzed. The joint analysis over weeks is not suggested because the same randomization plan was followed week after week. The particular pattern of taking observations may have some effect upon the results, and this would introduce a correlation into the observations from week to week analysis.

Figure 1. Sixty-four treatments used in studying rate of movement

Level of factor			
a b'c'd'e'	a b'c'd'e'	a b'c'd'e'	a b'c'd'e'
0 0 0 0 0	0 2 0 2 0	1 0 0 0 1	1 2 0 2 1
0 0 0 0 2	0 2 0 2 2	1 0 0 0 3	1 2 0 2 3
0 0 1 2 1	0 2 1 0 1	1 0 1 2 0	1 2 1 0 0
0 0 1 2 3	0 2 1 0 3	1 0 1 2 2	1 2 1 0 2
0 0 2 1 1	0 2 2 3 1	1 0 2 1 0	1 2 2 3 0
0 0 2 1 3	0 2 2 3 3	1 0 2 1 2	1 2 2 3 2
0 0 3 3 0	0 2 3 1 0	1 0 3 3 1	1 2 3 1 1
0 0 3 3 2	0 2 3 1 2	1 0 3 3 3	1 2 3 1 3
0 1 0 1 1	0 3 0 3 1	1 1 0 1 0	1 3 0 3 0
0 1 0 1 3	0 3 0 3 3	1 1 0 1 2	1 3 0 3 2
0 1 1 3 0	0 3 1 1 0	1 1 1 3 1	1 3 1 1 1
0 1 1 3 2	0 3 1 1 2	1 1 1 3 3	1 3 1 1 3
0 1 2 2 0	0 3 2 0 0	1 1 2 2 1	1 3 2 0 1
0 1 2 2 2	0 3 2 0 2	1 1 2 2 3	1 3 2 0 3
0 1 3 0 1	0 3 3 2 1	1 1 3 0 0	1 3 3 2 0
0 1 3 0 3	0 3 3 2 3	1 1 3 0 2	1 3 3 2 2

a_0 = large store
a_1 = small store

b_0 = cities over 100,000
b_1 = cities between 20,000 and 100,000
b_2 = cities between 5,000 and 20,000
b_3 = cities under 5,000

c_0 = Buffalo area
c_1 = Binghamton area
c_2 = Syracuse area
c_3 = Rochester area

d_0 = Monday, d_1 = Tuesday, d_2 = Wednesday, d_3 = Thursday — For first part of week

e_0 = 8 a.m. to 10 a.m., e_1 = 10 a.m. to noon, e_2 = noon to 2 p.m., e_3 = 2 p.m. to 4 p.m. — For first part of week

Figure 2. Sixty-four treatments renumbered as a 2^9 factorial

abc def ghi		abc def ghi	
000 000 000	(000 000 100)	100 000 010	(100 000 110)
000 000 001	(000 000 101)	100 000 011	(100 000 111)
000 100 110	(000 100 010)	100 100 100	(100 100 000)
000 100 111	(000 100 011)	100 100 101	(100 100 001)
000 011 010		100 011 000	
000 011 011		100 011 001	
000 111 100		100 111 110	
000 111 101		100 111 111	
010 001 010		110 001 000	
010 001 011		110 001 001	
010 101 100		110 101 110	
010 101 101		110 101 111	
010 010 100		110 010 110	
010 010 101		110 010 111	
010 110 010		110 110 000	
010 110 011		110 110 001	
001 000 100	(001 000 000)	101 000 110	(101 000 010)
001 000 101	(001 000 001)	101 000 111	(101 000 011)
001 100 010	(001 100 110)	101 100 000	(101 100 100)
001 100 011	(001 100 111)	101 100 001	(101 100 101)
001 011 110		101 011 100	
001 011 111		101 011 101	
001 111 000		101 111 010	
001 111 001		101 111 011	
011 001 110		111 001 100	
011 001 111		111 001 101	
011 101 000		111 101 010	
011 101 001		111 101 011	
011 010 000		111 010 010	
011 010 001		111 010 011	
011 110 110		111 110 100	
011 110 111		111 110 101	

Figure 3. Aliases in a one-eighth replicate of a 2^9 factorial

Defining contrast								Number of effect
	I	BEF**	ADFH**	ABDEH**	ACGH*	ABCEFGH *	CDFG*	BCDEG *	1
Main Effects	A	ABEF	DFH	BDEH	CGH	BCEFGH	ACDFG	ABCDEG	2
	B	EF	ABDFH	ADEH	ABCGH*	ACEFGH *	BCDFG *	CDEG*	3
	C	BCEF	ACDFH	ABCDEH	AGH	ABEFGH	DFG	BDEG	4
	BC	CEF	ABCDFH	ACDEH	ABGH	AEFGH	BDFG	DEG	5
	D	BDEF	AFH	ABEH	ADCGH	ABCDEFGH	CFG	BCEG	6
	E	BF	ADEFH	ABDH	ACEGH*	ABCFGH *	CDEFG*	BCDG *	7
	DE	BDF	AEFH	ABH	ACDEGH	ABCDFGH	CEFG	BCG	8
	F	BE	ADH	ABDEFH	ACFGH *	ABCEGH*	CDG*	BCDEFG*	9
	G	BEFG	ADFGH	ABDEGH	ACH	ABCEFH	CDF	BCDE	10
	FG	BEG	ADGH	ABDEFGH	ACFH	ABCEH	CD	BCDEF	11
	H	BEFH	ADF	ABDE	ACG	ABCEFG	CDFG	BCDEGH	12
	I	BEFI	ADFHI	ABDEHI	ACGHI	ABCEFGHI	CDFGI	BCDEGI	13
	HI	BEFHI	ADFI	ABDEI	ACGI	ABCEFGI	CDFGHI	BCDEGHI	14
Size store x Size town	AB	AEF	BDFH	DEH	BCGH	CEFGH	ABCDFG	ACDEG	15
	AC	ABCEF	CDFH	BCDEH	GH	BEFGH	ADFG	ABDEG	16
	ABC	ACEF	BCDFH	CDEH	BGH	EFGH	ABDFG	ADEG	17
Size store x Day of week	AF	ABE	DH	BDEFH	CFGH	BCEGH	ACDG	ABCDEFG	18
	AG	ABEFG	DFGH	BDEGH	CH	BCEFH	ACDF	ABCDE	19
	AFG	ABEG	DGH	BDEFGH	CFH	BCEH	ACD	ABCDEF	20
Size store x Time of day	AH	ABEFH	DF	BDE	CG	BCEFG	ACDFGH	ABCDEGH	21
	AI	ABEFI	DFHI	BDEHI	CGHI	BCEFGHI	ACDFGI	ABCDEGI	22
	AHI	ABEFHI	DFI	BDEI	CGI	BCEFGI	ACDFGHI	ABCDEGHI	23
Size town x Geographic area	BD	DEF	ABFH	AEH	ABCDGH	ACDEFGH	BCFG	CEG	24
	BE	see F							
	BDE	see AH							
	CD	see FG							
	CE	BCF	ACDEFH	ABCDH	AEGH	ABFGH	DEFG	BDG	25
	CDE	BCDF	ACEFH	ABCH	ADEGH	ABDFGH	EFG	BG	26
	BCD	CDEF	ABCFH	ACEH	ABDGH	ADEFGH	BFG	EG	27
	BCE	CF	ABCDEFH	ACDH	ABEGH	AFGH	BDEFG	DG	28
	BCDE	see G							

* Ratio of pluses to minuses either 48:16 or 16:48

** Ratio of pluses to minuses either ~~48~~:0 or 0:~~48~~
64 64

Figure 3, continued

Size town x Time of day	BH	EFH	ABDF	ADE	ABCG	ACEFG	BCDFGH	CDEGH	29
	BI	EFI	ABDFHI	ADEHI	ABCGHI	ACEFGHI	BCDFGI	CDEGI	30
	BHI	EFHI	ABDFI	ADEI	ABCGI	ACEFGI	BCDFGHI	CDEGHI	31
	CH see AG								
	CI	BCEFI	ACDFHI	ABCDEHI	AGHI	ABEFGHI	DFGI	BDEGI	32
	CHI	BCEFHI	ACDFI	ABCDEI	AGI	ABEFGI	DFGHI	BDEGHI	33
	BCH	CEFH	ABCDF	ACDE	ABG	AEFG	BDFGH	DEGH	34
	BCI	CEFI	ABCDFHI	ACDEHI	ABGHI	AEFGHI	BDFGI	DEGI	35
	BCHI	CEFHI	ABCDFI	ACDEI	ABGI	AEFGI	BDFGHI	DEGHI	36
Size store x Geographic area	AD	ABDEF	FH	BEH	CDGH	BCDEFGH	ACFG	ABCEG	37
	AE	ABF	DEFH	BDH	CEGH	BCFGH	ACDEFG	ABCDG	38
	ADE see BH								
Size city x Day of week	BF see E								
	BG see CDE								
	BFG see BCD								
	CF see BCE								
	CG see BDE								
	CFG see D								
	BCF see CE								
	BCG see DE								
	BCFG see BD								
Geographic area x Day of week	DF see CG								
	DG see CF								
	DFG see C								
	EF see B								
	EG see BFG								
	EFG see BG								
	DEF see BCFG								
	DEG see BC								
	DEFG see BCF								
Geographic area x Time of day	DH see AF								
	DI	BDEFI	AFHI	ABEHI	ACDGHI	ABCDEFGHI	CFGI	BCEGI	39
	DHI	BDEFHI	AFI	ABEI	ACDGI	ABCDEFGI	CFGHI	BCEGHI	40
	EH	BFH	ADEF	ABD	ACEG	ABCFG	CDEFGH	BCDGH	41
	EI	BFI	ADEFHI	ABDHI	ACEGHI	ABCFGHI	CDEFGI	BCDGI	42
	EHI	BFHI	ADEFI	ABDI	ACEGI	ABCFGI	CDEFGHI	BCDGHI	43
	DEH see AB								
	DEI	BDFI	AEFHI	ABHI	ACDEGHI	ABCDFGHI	CEFGI	BCGI	44
	DEHI	BDFHI	AEFI	ABI	ACDEGI	ABCDFGI	CEFGHI	BCGHI	45

Figure 3, continued

Day of week x Time of day	FH	see AD						
	FI	BEI	ADHI	ABDEFHI	ACFGHI	ABCEGHI	CDGI	BCDEFGI	46
	FHI	BEHI	ADI	ABDEFI	ACFGI	ABCEGI	CDGHI	BCDEFGHI	47
	GH	see AC						
	GI	BEFGI	ADFGHI	ABDEGHI	ACHI	ABCEFHI	CDFI	BCDEI	48
	GHI	BEFGHI	ADFGI	ABDEGI	ACI	ABCEFI	CDFHI	BCDEHI	49
	FGH	BEGH	ADG	ABDEFG	ACF	ABCE	CDH	BCDEFH	50
	FGI	BEGI	ADGHI	ABDEFGHI	ACFHI	ABCEHI	CDI	BCDEFI	51
	FGHI	BEGHI	ADGI	ABDEFGI	ACFI	ABCEI	CDHI	BCDEFHI	52
	AEG	ABFG	DEFGH	BDGH	CEH	BCFH	ACDEF	ABCD	53
	AEI	ABFI	DEFHI	BDHI	CEGHI	BCFGHI	ACDEFGI	ABCDGI	54
	ACE	ABCF	CDEFH	BCDH	CGH	BFGH	ADEFG	ABDG	55
	AFGI	ABEGI	DGHI	BDEFGHI	CFHI	BCEHI	ACDI	ABCDEFI	56
	AEHI	ABFHI	DEFI	BDI	CEGI	BCFGI	ACDEFGHI	ABCDGHI	57
	AEGI	ABFGI	DEFGHI	BDGHI	CEHI	BCFHI	ACDEFI	ABCDI	58
	ACEI	ABCFI	CDEFHI	BCDHI	EGHI	BFGHI	ADEFGI	ABDGI	59
	ABCI	ACEFI	BCDFHI	CDEHI	BGHI	EFGHI	ABDFGI	ADEGI	60
	AEGHI	ABFGHI	DEFGI	BDGI	CEI	BCFI	ACDEFHI	ABCDHI	61
	ACDHI	ABCDEFHI	CFI	BCEI	DGI	BDEFGI	AFGHI	ABEGHI	62
	ACDEG	see DEH						
	ACEHI	ABCFHI	CDEFI	BCDI	EGI	BFGI	ADEFGHI	ABDGHI	63
	ABCHI	ACEFHI	BCDFI	CDEI	BGI	EFGI	ABDFGHI	ADEGHI	64

Figure 4. Analysis of variance for the fractional replicate in figure 1

Source of variation		df	
Size of store	A	1	
Size of town	B	1	3
	C	1	
	BC	1	
Geographic area	D	1	3
	E	1	
	DE	1	
Day of week	F (alias of BE)	1	3
	G (alias of BCDE)	1	
	FG (alias of CD)	1	
Time of day	H	1	3
	I	1	
	HI	1	
Size of store x Size of town	AB	1	3
	AC	1	
	ABC	1	
Size of store x Day of week	AF	1	3
	AG (alias of CH)	1	
	AFG	1	
Size of store x Time of day	AH (alias of BDE)	1	3
	AI	1	
	AHI	1	
Size of town x Geographic area	BD	1	5
	BE (alias of F)	0	
	BDE (alias of AH)	0	
	CD (alias of FG)	0	
	CE	1	
	CDE	1	
	BCD	1	
	BCE	1	
	BCDE (alias of G)	0	
Size of town x Time of day	BH	1	8
	BI	1	
	BHI	1	
	CH (alias of AG)	0	
	CI	1	
	CHI	1	
	BCH	1	
	BCI	1	
	BCHI	1	
Residual			28
Total			63